

On isomorphisms of algebras of compactly supported continuous functions

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ABSTRACT. We study the general form of isomorphisms on the algebra of compactly supported complex-valued continuous functions defined on a locally compact Hausdorff space (the proof of which works for the algebra of $\mathcal{C}^{(k)}$ -differentiable functions on a $\mathcal{C}^{(k)}$ -manifold as well). We obtain using only topological techniques, that any such map is a composition of a homeomorphism of the locally compact spaces (resp. $\mathcal{C}^{(k)}$ -diffeomorphism), and an automorphism of the field of complex numbers. In the particular case when X is a locally compact group, and the map preserves convolution products, the resulting homeomorphism is also a group isomorphism. An application of this gives a characterisation of the Fourier transform on the algebra of Schwartz-Bruhat functions on locally compact Abelian groups.

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1. Introduction

The fact that isomorphisms of algebras of functions defined over certain topological spaces X , taking values in a field \mathbb{F} determine the space X have been known for long since the work of Gelfand and Kolmogoroff[5] and Milgram[8]. In 2005, Mrčun[9] proved the following result for isomorphisms of algebras of smooth functions. We denote $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .

THEOREM 1.1. ([9]) *For any Hausdorff smooth manifolds M and N (which are not necessarily second-countable, paracompact or connected), any isomorphism of algebras of smooth functions $T : \mathcal{C}^\infty(N, \mathbb{F}) \rightarrow \mathcal{C}^\infty(M, \mathbb{F})$ is given by composition with a unique diffeomorphism $\tau : M \rightarrow N$.*

Mrčun also proved an analogous result when the spaces $\mathcal{C}^\infty(N, \mathbb{F})$ and $\mathcal{C}^\infty(M, \mathbb{F})$ are replaced by $\mathcal{C}_c^\infty(N, \mathbb{F})$ and $\mathcal{C}_c^\infty(M, \mathbb{F})$ respectively.

J. Grabowski([6]) studied the above question for unital algebras of functions on more general topological spaces.

For a function $f : X \rightarrow \mathbb{F}$, we denote by $\text{supp } f$, the support of f , defined by $\text{supp } f := \text{closure of } \{x \in X : f(x) \neq 0\}$.

DEFINITION 1.2. ([6]) *For a topological space X , a subalgebra \mathfrak{S} of $\mathcal{C}(X, \mathbb{F})$ is called distinguishing if the following conditions hold good:*

- (1) *If $f \in \mathfrak{S}$ is nowhere vanishing, then $f^{-1} \in \mathfrak{S}$,*
- (2) *for each $p \in X$, and every open neighbourhood U of p , there exists $g \in \mathfrak{S}$ such that $g(X) \subseteq [0, 1]$, $\text{supp}(g) \subset U$, and $g(p) = 1$ if and only if $p' = p$.*

THEOREM 1.3. ([6]) *Let \mathfrak{S}_i be a distinguishing algebra of \mathbb{F} -valued continuous functions on a topological space X_i , $i = 1, 2$. Then, every algebra isomorphism $\Phi : \mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ is the pullback by a homeomorphism $\varphi : X_2 \rightarrow X_1$.*

In [6], the above result is initially proved for the algebra $\mathcal{C}(X, \mathbb{F})$, and then deduced for a distinguishing subalgebra. The former result is proved using the maximal ideal space theory for unital Banach algebras. In the present paper, we obtain using topological techniques, the explicit form of the algebra isomorphism, first for the algebra $\mathcal{C}_c(X, \mathbb{F})$ (which is not unital unless X is compact). The corresponding result for the algebra $\mathcal{C}_c^{(k)}$ — functions on a $\mathcal{C}^{(k)}$ — manifold follows as a consequence. Surprisingly, the proof of this result does not make any use of the differentiable structure on the underlying manifold.

Before stating our result, we discuss a few more related results.

In [2], the authors obtained the general form of a multiplicative bijection between certain classes \mathcal{B} of \mathbb{F} -valued functions.

THEOREM 1.4. ([2]) *Let M be a topological real manifold and \mathcal{B} be either $\mathcal{C}(M, \mathbb{C})$ or $\mathcal{C}_c(M, \mathbb{C})$. Let $T : \mathcal{B} \rightarrow \mathcal{B}$ be a multiplicative bijection. Then there exists some homeomorphism $u : M \rightarrow M$ and a function $p \in \mathcal{C}(M, \mathbb{C})$, $\Re(p) > 0$, such that either $T(re^{i\theta})(u(x)) = |r(x)|^{p(x)} e^{i\theta(x)}$ or $T(re^{i\theta})(u(x)) = |r(x)|^{p(x)} e^{-i\theta(x)}$.*

A similar result holds good for the algebras \mathcal{C}^k of k -times differentiable functions as well.

THEOREM 1.5. ([2]) *Let M be a \mathcal{C}^k real manifold, $1 \leq k \leq \infty$, and \mathcal{B} be one of the following function spaces: $\mathcal{C}^k(M, \mathbb{C})$, $\mathcal{C}_c^k(M, \mathbb{C})$ or $\mathcal{S}_{\mathbb{C}}(n)$. Let $T : \mathcal{B} \rightarrow \mathcal{B}$ be a multiplicative bijection. Then there exists some \mathcal{C}^k -diffeomorphism $u : M \rightarrow M$ such that either $Tf(u(x)) = f(x)$ or $Tf(u(x)) = \overline{f(x)}$.*

The fundamental difference between the above results and the present one is that of the spaces on which the function algebras are defined. In the former case, these are differentiable manifolds. Hence the proofs also involved ideas on jet bundles of functions which are not available in the current setting of locally compact Hausdorff spaces. Also, the algebra of smooth functions is replaced with that of compactly supported continuous functions defined on a locally compact Hausdorff space. However, the proof of Theorem 1.6 gives a *unified* proof for the description of isomorphisms on the algebras of $\mathcal{C}_c^{(k)}$ -differentiable functions on k -differentiable manifolds, for all $1 \leq k \leq \infty$.

We denote by $\mathcal{C}_c(+, \cdot, *)$, the algebra of all compactly supported complex-valued functions on X , under pointwise addition and multiplication.

Our main result is

THEOREM 1.6. *Let X and Y be locally compact Hausdorff spaces. Let $T : (\mathcal{C}_c(X), +, \cdot) \rightarrow (\mathcal{C}_c(Y), +, \cdot)$ be an isomorphism of algebras. Then there exists a homeomorphism $\psi : Y \rightarrow X$ such that either $Tf = f \circ \psi$ for all $f \in \mathcal{C}_c(X)$, or $Tf = \overline{f \circ \psi}$ for all $f \in \mathcal{C}_c(X)$.*

REMARK 1.1. Apriori, the isomorphism is defined only on functions in $\mathcal{C}_c(X, \mathbb{F})$. But as illustrated in the proof of the above theorem, the isomorphism has a canonical extension to the algebra of constant functions. This extension yields the explicit form of the isomorphism using ideas only from point-set topology.

In order to prove Theorem 1.6, first we study the general structure of the map T .

For $x_0 \in X$, define

$$S(x_0) := \{f \in \mathcal{C}_c(X) : x_0 \in \text{Supp } f\}.$$

PROPOSITION 1.7. *Let X and Y be locally compact Hausdorff spaces. Let $T : \mathcal{C}_c(X) \rightarrow \mathcal{C}_c(Y)$ be a multiplicative bijection. For any $x_0 \in X$, there exists $y_0 \in Y$ such that $Tf \in S(y_0)$ whenever $f \in S(x_0)$.*

PROOF. First, we observe that for functions $f, g \in \mathcal{C}_c(X)$, we have $Tg = 1$ on $\text{supp } Tf$ whenever $g = 1$ on $\text{supp } f$.

The condition $g = 1$ on $\text{supp } f$ gives $f \cdot g = f$. Since T preserves products, this gives $Tf \cdot Tg = T(f \cdot g) = Tf$, which guarantees $Tg = 1$ wherever Tf is nonzero.

Let $y_0 \in \text{supp } Tf$ be such that $Tf(y_0) = 0$. This gives a net (y_τ) converging to y_0 such that $Tf(y_\tau) \neq 0$ for any τ . By the above argument, $Tg(y_\tau) = 1$ for all τ , and hence $Tg(y_0) = 1$.

Let $E := \{f \in \mathcal{C}_c(X) : f(x_0) \neq 0\}$.

Fix $g \in E$. Then $K := \text{supp}(Tg)$ is compact. For $f \in E$, let $K(f) := K \cap \text{supp}(Tf)$. For any finite collection $\{f_0 := g, f_1, \dots, f_k\}$ in E , the product $\prod_{j=0}^k f_j \neq 0$. This gives by the multiplicativity of the map T , that $T(\prod_{j=0}^k f_j) = \prod_{j=0}^k Tf_j \neq 0$. This ensures $\cap_{j=0}^k K_{f_j} \neq \emptyset$. In otherwords, the collection $\{K_f : f \in E\}$ of closed subsets of the compact set K satisfies finite intersection property. This guarantees the existence of some element $y_0 \in \bigcap_{f \in E} K_f \neq \emptyset$.

Claim. For any function $f \in \mathcal{C}_c(X)$, we have $Tf \in S(y_0)$ whenever $f \in S(x_0)$.

Proof of Claim. We prove the claim in two cases.

Case 1. $f(x_0) \neq 0$.

Choose $g \in \mathcal{C}_c(X)$ with $f \cdot g = 1$ on a neighbourhood V of x_0 . Let $h \in \mathcal{C}_c(X)$ be such that $h = 1$ on a neighbourhood W of x_0 and $\text{supp } h \subseteq V$. Now, $f \cdot g = 1$ on $\text{supp } h$ gives $Tf \cdot Tg = 1$ on $\text{supp } Th$, which contains y_0 . Thus $Tf \in S(y_0)$.

Case 2. $f(x_0) = 0$.

Suppose $Tf \notin S(y_0)$. Let W be a neighbourhood of y_0 on which Tf vanishes identically. Choose $h \in \mathcal{C}_c(Y)$ with $\text{supp } h \subseteq W$ and $h(y_0) \neq 0$. For $g \in \mathcal{C}_c(X)$ with $Tg = h$, we have

$$0 \equiv Tf \cdot h = Tf \cdot Tg = T(f \cdot g).$$

This means $f \cdot g \equiv 0$, which is not possible as $g(x_0) \neq 0$ and also $x_0 \in \text{supp } f$. □

Let $\varphi : X \rightarrow Y$ be defined as follows:

$$\varphi(x_0) = y_0, \quad \text{if } Tf \in S(y_0) \text{ for all } f \in S(x_0).$$

PROPOSITION 1.8. *The map $\varphi : X \rightarrow Y$ is a homeomorphism.*

PROOF. First we ensure that the map is well-defined. Suppose there exists $x_0 \in X$ with $\varphi(x_0) = y_1$, $\varphi(x_0) = y_2$ and $y_1 \neq y_2$. Let $g_1, g_2 \in \mathcal{C}_c(Y)$ be supported in disjoint neighbourhoods V_1 and V_2 of y_1 and y_2 , respectively, with $g_1(y_1) \neq 0$ and $g_2(y_2) \neq 0$. Then for f_1 and f_2 such that $Tf_1 = g_1$ and $Tf_2 = g_2$, we have

$$0 \equiv g_1 \cdot g_2 = Tf_1 \cdot Tf_2 = T(f_1 \cdot f_2),$$

and hence $f_1 \cdot f_2 \equiv 0$. This is in contradiction with $(f_1 \cdot f_2)(x_0) \neq 0$.

Repeating the above arguments for T^{-1} instead of T gives that φ is a bijection.

Suppose φ is not continuous at some point x_0 . Let (x_τ) be a net converging to $x_0 \in X$ with $\varphi(x_\tau)$ not converging to $\varphi(x_0)$. Let $h \in \mathcal{C}_c(Y)$ be such that $h(\varphi(x_0)) = 1$ and $\text{supp } h \subseteq V$, for a neighbourhood V of $\varphi(x_0)$ which does not contain $\varphi(x_\tau)$ for any τ . For the function $g \in \mathcal{C}_c(X)$ such that $Tg = h$, we have $g(x_\tau) = 0$ for all τ , and hence $g(x_0) = 0$, which contradicts $Tg(\varphi(x_0)) = 1$. A similar argument for T^{-1} gives that T is a homeomorphism. □

We are now ready to prove Theorem 1.6.

PROOF OF THEOREM 1.6. By Propositions 1.7 and 1.8, we infer that the spaces X and Y are homeomorphic.

Though the map T is apriori defined only on $\mathcal{C}_c(X)$, one can extract information as to how T acts on the constant functions on X , which we denote just by the constant itself.

For $f, g \in \mathcal{C}_c(X)$, and $\alpha (\neq 0) \in \mathbb{C}$, we have

$$T(\alpha f)(y) Tg(y) = T(\alpha fg)(y) = T(f)(y) T(\alpha g)(y), \quad y \in Y.$$

For $y \in Y$, let $h \in \mathcal{C}_c(X)$ be such that $Th(y) \neq 0$. Then we have

$$\begin{aligned} T(\alpha f)(y) &= \frac{T(\alpha h)(y)}{Th(y)} Tf(y) \text{ for all } f \in \mathcal{C}_c(X) \\ &= m(\alpha, y) Tf(y) \text{ (say).} \end{aligned}$$

Thus $T(\alpha f)(y) = m(\alpha, y) Tf(y)$, for all $y \in Y$. By definition, the function $m(\cdot, \cdot)$ is continuous in the second variable, as a function of $y \in Y$.

For $y \in Y$, choose $f \in \mathcal{C}_c(Y)$ such that $Tf(y) \neq 0$. Then

$$Tf(y) = T(1 \cdot f)(y) = m(1, y)Tf(y).$$

Varying y over Y and appropriately varying f , we get $m(1, \cdot) \equiv 1$.

A similar argument gives $m(0, \cdot) \equiv 0$.

For $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{C}_c(X)$ such that $Tf(y) \neq 0$, we have

$$m(\alpha + \beta, y)Tf(y) = T((\alpha + \beta)f)(y) = [m(\alpha, y) + m(\beta, y)]Tf(y).$$

Also,

$$m(\alpha\beta, y)Tf(y) = T(\alpha\beta f)(y) = m(\alpha, y)T(\beta f)(y) = m(\alpha, y)m(\beta, y)Tf(y).$$

Thus $m(\cdot, \cdot)$ is additive and multiplicative in the first variable.

Suppose $f \in \mathcal{C}_c(X)$ with $f = c(\neq 0)$ near x_0 . Then $\frac{f}{c} = 1$ near x_0 and so $Tf(\cdot) = m(c, \cdot)$ near $y_0 (= \varphi(x_0))$. In particular, we have

$$Tf(y_0) = m(f(x_0), y_0).$$

Next, to find the action of T on a general function in $\mathcal{C}_c(X)$. Fix $g \in \mathcal{C}_c(X)$ and $x_0 \in X$.

If $g(x_0) = 0$, then we have

$$Tg(y_0) = Tg(\varphi(x_0)) = 0 = m(0, y_0).$$

Suppose $g(x_0) \neq 0$. For a function $f \in \mathcal{C}_c(X)$ with $f = g(x_0)$ near x_0 , we have

$$0 = T(f - g)(y_0) = Tf(y_0) - Tg(y_0) = m(f(x_0), y_0) - Tg(y_0) = m(g(x_0), y_0) - Tg(y_0).$$

Working with appropriate locally constant functions f , we get

$$(1.1) \quad Tg[\varphi(x)] = m(g(x), \varphi(x)) \text{ for all } g \in \mathcal{C}_c(X).$$

Since all the other functions involved in the above equation are continuous, we get $m(\cdot, \cdot)$ is continuous in both the variables.

We have so far observed that for any $y \in Y$, the map $m(\cdot, y)$ is a continuous additive and multiplicative automorphism of \mathbb{C} , and hence we get $m(\alpha, y) = \alpha$ or $m(\alpha, y) = \bar{\alpha}$ for all $\alpha \in \mathbb{C}$. Since $m(\cdot, \cdot)$ is continuous in the second variable, we are left with $m(\alpha, \cdot) \equiv \alpha$ or $m(\alpha, \cdot) \equiv \bar{\alpha}$.

By definition, the map $m(\cdot, \cdot)$ is canonical to the map T . Hence by abuse of notation, we may define $T(\alpha) := m(\alpha, y_0)$ for $\alpha \in \mathbb{C}$, and a fixed $y_0 \in Y$. Here we emphasise that apriori, the map T was defined on $\mathcal{C}_c(X)$, which does not contain the constant functions unless X is compact.

Combining the above observations with Equation (1.1), we get

$$Tf[\varphi(x)] = T(f(x)), \quad \text{for all } f \in \mathcal{C}_c(X).$$

Since $\psi = \varphi^{-1}$, we get that either $Tf(x) = f[\psi(x)]$ or $Tf(x) = \overline{f[\psi(x)]}$. \square

The proof of the above result provides a unified proof of the description of algebra isomorphisms of $\mathcal{C}^{(k)}$ –functions on a real $\mathcal{C}^{(k)}$ –differentiable manifold, for $0 \leq k \leq \infty$.

COROLLARY 1.9. *Let M and N be real $\mathcal{C}^{(k)}$ –manifolds for some k , $0 \leq k \leq \infty$. Let $T : (\mathcal{C}_c^{(k)}(M), +, \cdot) \rightarrow (\mathcal{C}_c^{(k)}(N), +, \cdot)$ be an isomorphism of algebras. Then there exists a $\mathcal{C}^{(k)}$ –diffeomorphism $\psi : N \rightarrow M$ such that either $Tf = f \circ \psi$ for all $f \in \mathcal{C}_c^{(k)}(M)$, or $Tf = \overline{f \circ \psi}$ for all $f \in \mathcal{C}_c^{(k)}(M)$.*

PROOF. Follows from the proof of Theorem 1.6. The conclusion that the map ψ is a $\mathcal{C}^{(k)}$ –diffeomorphism follows from the explicit form of the algebra isomorphism. \square

Theorem 1.6 proves that the topology of X is completely determined by the algebraic structure of $(\mathcal{C}_c(X), +, \cdot)$. It is natural to ask if $\mathcal{C}_c(X)$ carries any information on the *algebraic* structure of X .

Suppose X is a locally compact group. Then there is a unique left Haar measure μ on X . For functions $f, g \in \mathcal{C}_c(X)$, the convolution product $f * g$ is defined as

$$f * g(x) = \int_G f(x - y) g(y) d\mu(y).$$

Equipped with the convolution product, the space $\mathcal{C}_c(X)$ forms an algebra, which we denote by $(\mathcal{C}_c(X), +, *)$. Theorem 1.6 states that the algebra $(\mathcal{C}_c(X), +, \cdot)$ determines the topology of X .

It is natural to ask the following questions:

- (1) Does $\mathcal{C}_c(X)$ also determine the algebraic structure of X ?
- (2) Does the explicit form of such an isomorphism depend on the structure of the field \mathbb{F} ?

Our next result gives a positive answer to the above questions in the particular cases when $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .

THEOREM 1.10. *Let G and H be locally compact groups. Let $T : \mathcal{C}_c(G) \rightarrow \mathcal{C}_c(H)$ be an algebra isomorphism under pointwise and convolution products. Then there exists a multiplicative homeomorphism $\psi : H \rightarrow G$ such that either $Tf(y) = f(\psi(y))$ for all $f \in \mathcal{C}_c(G)$, or $Tf(y) = \overline{f(\psi(y))}$ for all $f \in \mathcal{C}_c(G)$.*

PROOF. In view of Theorem 1.6, it remains to prove that the map $\psi^{-1} = \varphi : G \rightarrow H$ is a homomorphism.

Suppose $\varphi(xy) \neq \varphi(x)\varphi(y)$ for some $x, y \in G$.

We denote $(\varphi \otimes \varphi)(x, y) = \varphi(x)\varphi(y)$, $x, y \in G$. Let p_G and p_H denote the product maps of the respective groups.

Suppose $(\varphi \circ p_G)(x, y) \neq [p_H \circ (\varphi \otimes \varphi)](x, y)$ for some $x, y \in G$. Let V_{xy} and $V_{x \diamond y}$ be disjoint neighbourhoods of $(\varphi \circ p_G)(x, y)$ and $[p_H \circ (\varphi \otimes \varphi)](x, y)$, respectively. Since the maps $(\varphi \circ p_G)$ and $p_H \circ (\varphi \otimes \varphi)$ are continuous on $G \times G$, we get neighbourhoods $W_{x,1}$, $W_{y,1}$ and $W_{xy,1}$ of x , y and xy , respectively, such that

$$\varphi(W_{xy,1}) \subseteq V_{xy} \quad \text{and} \quad W_{x,1}W_{y,1} \subseteq W_{xy,1}.$$

Hence
$$\varphi(W_{x,1}W_{y,1}) \subseteq \varphi(W_{xy,1}) \subseteq V_{xy}.$$

Similarly, continuity of the map $[p_H \circ (\varphi \otimes \varphi)]$, gives rise to neighbourhoods $W_{x,2}$, $W_{y,2}$, $V_{x,2}$ and $V_{y,2}$ of x , y , $\varphi(x)$ and $\varphi(y)$, respectively, such that

$$V_{x,2} V_{y,2} \subseteq V_{x \diamond y}, \quad \varphi(W_{x,2}) \subseteq V_{x,2} \quad \text{and} \quad \varphi(W_{y,2}) \subseteq V_{y,2}.$$

Let $W_x = W_{x,1} \cap W_{x,2}$, $W_y = W_{y,1} \cap W_{y,2}$. We have

$$\begin{aligned}
 \varphi(W_x) &\subseteq \varphi(W_{x,2}) \subseteq V_{x,2} = V_x \text{ (say)} \\
 \varphi(W_y) &\subseteq \varphi(W_{y,2}) \subseteq V_{y,2} = V_y \text{ (say)} \\
 (1.2) \quad \varphi(W_x W_y) &\subseteq \varphi(W_{x,1} W_{y,1}) \subseteq \varphi(W_{xy,1}) \subseteq V_{xy}
 \end{aligned}$$

Let nonzero functions $f_x, f_y \in \mathcal{C}_c(G)$ be such that $\text{supp } f_x \subseteq W_x$ and $\text{supp } f_y \subseteq W_y$. Let $g_x = Tf_x$ and $g_y = Tf_y$. Then $T(f_x * f_y) = g_x * g_y \neq 0$.

We have

$$\begin{aligned}
 \text{supp}(g_x * g_y) &= \text{supp } T(f_x * f_y) \subseteq (\text{supp } Tf_x) (\text{supp } Tf_y) \\
 &= \varphi(\text{supp } f_x) \varphi(\text{supp } f_y) \subseteq \varphi(W_x) \varphi(W_y) \\
 (1.3) \quad &\subseteq V_{x,2} V_{y,2} \subseteq V_{x \diamond y}
 \end{aligned}$$

But $\text{supp } (f_x * f_y) \subseteq (\text{supp } f_x) (\text{supp } f_y) \subseteq W_x W_y$. By (1.2), this gives

$$\begin{aligned}
 \text{supp}(g_x * g_y) &= \text{supp}(Tf_x * Tf_y) = \text{supp } T(f_x * f_y) \\
 (1.4) \quad &= \varphi(\text{supp}(f_x * f_y)) \subseteq \varphi(W_x W_y) \subseteq V_{xy}.
 \end{aligned}$$

From (1.3) and (1.4), we get

$$\text{supp}(g_x * g_y) \subseteq V_{x \diamond y} \cap V_{xy} = \emptyset.$$

This gives $g_x * g_y \equiv 0$, a contradiction to $f_x * f_y \neq 0$. Hence the map φ is a homomorphism. \square

COROLLARY 1.11. *In Theorem 1.10, in addition to the hypotheses, if $G = H$, then the map φ is measure-preserving.*

PROOF. We denote $\psi := \varphi^{-1}$. Suppose $Tf = f \circ \psi$ for all $f \in \mathcal{C}_c(G)$. For the identity element e of G , we have $\psi(e) = e$, and hence

$$\begin{aligned}
 (f * g)(e) &= T(f * g)(e) = (Tf * Tg)(e) \\
 &= \int_G Tf(y^{-1}) Tg(y) d\mu(y) \\
 &= \int_G f[\psi(y^{-1})] g[\psi(y)] d\mu(y) \\
 &= \int_G f[\psi(y)^{-1}] g[\psi(y)] d\mu(y) \\
 \text{i.e., } \int_G f(y^{-1}) g(y) d\mu(y) &= \int_G f(y^{-1}) g(y) d\mu(\varphi(y))
 \end{aligned}$$

For $f \in \mathcal{C}_c(G)$, choosing $g \in \mathcal{C}_c$ with $g = 1$ on $(\text{supp } f)^{-1}$, we get

$$\int_G f(y^{-1}) d\mu(y) = \int_G f(y^{-1}) d\mu(\varphi(y)).$$

Thus we have

$$\int_G f(y) d\mu(y) = \int_G (f \circ \psi)(y) d\mu(y)$$

for all functions $f \in \mathcal{C}_c(G)$. Hence the map ψ is measure-preserving.

A similar argument applies when T is of the form $Tf = \overline{f \circ \psi}$, proving our result. \square

We now recall the definition of the Schwartz-Bruhat space of functions on a locally compact Abelian group.

Let G be a locally compact Abelian group with unitary dual Γ . We denote by dx , the Haar measure on G . For an integrable function f on G , its Fourier transform is defined as

$$\widehat{f}(\xi) = \int_G f(x) \overline{\langle x, \xi \rangle} dx.$$

In the above, $\langle x, \xi \rangle$ stands for the dual action of $\xi \in \Gamma$ on $x \in G$.

For the function $f^*(x) = \overline{f(x^{-1})}$, we have $\widehat{[f^*]}(\xi) = [\widehat{f}(\xi)]^*$ for all $\xi \in \Gamma$.

F. Bruhat[3] extended the notion of a smooth function to a large class of groups, which encompasses the LCA groups. In 1975, M.S. Osborne characterised the Schwartz-Bruhat space of functions on a LCA group in terms of the asymptotic behavior of the function and its Fourier transform.

A function $f : G \rightarrow \mathbb{C}$ is said to belong the *Schwartz-Bruhat space* if f satisfies the following conditions:

- (a): $f \in \mathcal{C}^\infty(G)$.
- (b): $P(\partial)f \in L^\infty(G)$ for all polynomial differential operators $P(\partial)$, where the polynomial is in $\mathbb{R}^n \times \mathbb{Z}^k$ variables.

For any locally compact Abelian group G , we have

$$\mathcal{S}(G) = \lim_{\rightarrow} \mathcal{S}(H/K),$$

where the direct limit is taken over all pairs (H, K) of subgroups of G such that

- (i): The subgroup H is open and compactly generated,
- (ii): The subgroup K is compact
- (iii): The quotient H/K is a Lie group.

For a more detailed definition of the space $\mathcal{S}(G)$, we refer the reader to [7] and [13]. The following result gives a complete description of the Fourier transform on the Schwartz-Bruhat space of functions.

THEOREM 1.12. ([13]) *The Fourier transform maps $\mathcal{S}(G)$ isomorphically onto $\mathcal{S}(\Gamma)$ and $\mathcal{S}(G)$ is dense in $L^1(G)$.*

Let $\mathcal{C}_c^\infty = \mathcal{C}_c^\infty(G) := \{f \in \mathcal{S}(G) : \text{supp } f \text{ is compact}\}$.

The following results were obtained in [7]. However, we include short versions of their proofs here to illustrate how they are related to our main results.

THEOREM 1.13. ([7]) *Let G be a locally compact Abelian group, and Γ , its unitary dual. Let $T : \mathcal{S}(G) \rightarrow \mathcal{S}(\Gamma)$ be a bijection such that for all functions $f, g \in \mathcal{S}(G)$, we have*

- (a): $T(f + g^*) = T(f) + [T(g)]^*$,
- (b): $T(f \cdot g) = T(f) * T(g)$,
- (c): $T(f * g) = T(f) \cdot T(g)$.

Then there exists a measure-preserving multiplicative homeomorphism ψ of G onto itself such that either $Tf = \widehat{(f \circ \psi)}$ for all $f \in \mathcal{S}(G)$, or $Tf = \widehat{(f \circ \psi)}$ for all $f \in \mathcal{S}(G)$.

PROOF. Let $f \in \mathcal{S}(G)$. As the Fourier transform maps $\mathcal{S}(G)$ bijectively onto $\mathcal{S}(\Gamma)$, we get that $Tf = \widehat{g}$ for a unique function $g \in \mathcal{S}(G)$. Define $Uf := g$. Then the map U maps $\mathcal{S}(G)$ bijectively onto itself, such that for all functions $f, g \in \mathcal{S}(G)$, we have

- (1) $U(f + \overline{g}) = U(f) + \overline{U(g)}$,
- (2) $U(f \cdot g) = U(f) \cdot U(g)$,
- (3) $U(f * g) = U(f) * U(g)$.

The conclusion is a consequence of the following result for the map U . □

THEOREM 1.14. ([7]) *Let G be a locally compact Abelian group. Let $U : \mathcal{S}(G) \rightarrow \mathcal{S}(G)$ be a bijection satisfying the following conditions for all functions $f, g \in \mathcal{S}(G)$:*

- (1) $U(f + \overline{g}) = U(f) + \overline{U(g)}$,

- (2) $U(f \cdot g) = U(f) \cdot U(g)$,
- (3) $U(f * g) = U(f) * U(g)$.

Then there exists a measure-preserving multiplicative homeomorphism $\psi : G \rightarrow G$ such that either $Uf = f \circ \psi$ for all $f \in \mathcal{S}(G)$, or $Uf = \overline{f} \circ \psi$ for all $f \in \mathcal{S}(G)$.

PROOF. For $x_0 \in G$, define

$$S(x_0) := \{f \in \mathcal{S}(G) : x_0 \in \text{supp } f\}.$$

Let $f, g \in \mathcal{S}(G)$. As in the beginning of the proof of Theorem 1.6, we have that $Ug = 1$ on $\text{supp}(Uf)$ whenever $g = 1$ on $\text{supp } f$.

Claim. If $f \in \mathcal{C}_c(G)$, then $Uf \in \mathcal{C}_c(G)$.

Proof of Claim. Let $f \in \mathcal{C}_c(G)$ with $f(x_0) \neq 0$. Let $g \in \mathcal{S}(G)$ be such that $g = 1$ on $\text{supp } f$. Then $Ug = 1$ on $\text{supp}(Uf)$. As $Ug \in \mathcal{S}(G)$, we get that $\text{supp}(Uf)$ is compact.

We observe that the proof of Corollary 1.11 applies to functions in $\mathcal{S}(G)$, thus proving the result. \square

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